Eur. Phys. J. B **47**, 239–243 (2005)
DOI: 10.1140/epjb/e2005-00321-2

THE EUROPEAN PHYSICAL JOURNAL B

A note on Edwards' hypothesis for zero-temperature Ising dynamics

F. Camia[a]

EURANDOM, P.O. Box 513, 5600 MB Eindhoven, The Netherlands

Received 5 March 2005
Published online 11 October 2005 –

Abstract. We give a simple criterion for testing a behavior à la Edwards in certain zero-temperature, ferromagnetic spin-flip dynamics and use it to show that the limiting distributions of those dynamics do not coincide with the uniform distribution over the blocked configurations of the dynamics. We provide explicit examples in dimension one and higher.

PACS. 05.70.Ln Nonequilibrium and irreversible thermodynamics – 64.60.My Metastable phases – 68.43.Mn Adsorption/desorption kinetics – 75.40.Gb Dynamic properties (dynamic susceptibility, spin waves, spin diffusion, dynamic scaling, etc.)

1 Introduction

In many physical systems the dynamics at low temperature or high density is so slow that the system is out of equilibrium at all practical time scales and ordinary thermodynamics does not apply. As a consequence, because of their practical as well as theoretical interest, non-standard thermodynamics have been proposed to describe out-of-equilibrium systems.

In the context of granular materials, Edwards proposed to compute thermodynamic quantities by means of a flat ensemble average over all the blocked configurations of grains with prescribed density, leading to a natural definition of configurational temperature (see [1–3]). In the absence of a derivation from first principles, this approach has to be tested with specific models and experiments (see, for example [4], and references therein).

Since its proposal, there have been various attempts to apply Edwards' idea to situations far beyond its original scope, in particular in the context of the slow relaxation dynamics of glassy materials (see, for example, [5,6]) and of the tapping of spin systems (see, for example, [7–9]). (The literature is very vast and we will not try to provide an exhaustive list of the relevant papers.)

In this note we revisit one of those attempts [10], namely looking at zero-temperature Ising dynamics, and give some novel results for two- and higher-dimensional models. Although the situation considered here and in [10] is completely different from the original situation considered by Edwards, it has been extensively investigated by various authors (see [11] and references therein) as one where a behavior à la Edwards might arise. In fact, numerical studies have shown that applying Edwards' hypothesis provides reasonable predictions for physical quantities, sometimes giving very good numerical accuracy (see, for example, [11]).

One version of the so called Edwards' hypothesis in the context of zero-temperature dynamics with many absorbing (or blocked) configurations consists in assuming that all the absorbing configurations are sampled by the dynamics with equal weights, which would imply that the limiting spin distribution under the dynamics is the uniform distribution over all the absorbing configurations. This turns out not to be the case in general, as already concluded in [10] for certain one-dimensional examples.

In Section 2 we revisit a constrained Glauber dynamics analyzed in [10], then in Section 3 we present a general criterion and provide new examples in higher dimension where the limiting distribution differs from the uniform one. The main novelty of this note lies in the approach, which is mathematically rigorous and allows us to treat models in dimension higher than one, where the methods of [10] do not easily apply (see the discussion at the end of [11]).

The models considered here belong to a class of zero-temperature spin-flip dynamics that have been much studied in recent years (see, for example [12–14]), in an attempt to understand both the behavior of the dynamics (e.g., the speed of relaxation) and the properties and distribution of the absorbing configurations (e.g., percolation properties). From this point of view, it is interesting to check the validity of the so called Edwards' hypothesis since it gives information about the limiting distribution

[a] e-mail: camia@eurandom.tue.nl

of such models. A slightly longer version of this note with some more details and a further application of the method is available online [15].

1.1 FKG inequality and Harris' theorem

We give here the tools needed to check Edwards' hypothesis for attractive (see the next paragraph), zero-temperature Ising dynamics. In the context of an Ising spin model on a lattice $\mathbb{L}$, we will call *increasing* an event $\mathcal{E}$ such that its indicator function $I_{\mathcal{E}}(\sigma)$ is non-decreasing in the number of plus spins present in the configuration $\sigma = \{\sigma_x\}_{x\in\mathbb{L}}$, $\sigma_x = \pm 1$. If $\mathcal{E}_1$ and $\mathcal{E}_2$ are two increasing events, the FKG inequality [16–18] states that, roughly speaking, the occurrence of $\mathcal{E}_2$ makes $\mathcal{E}_1$ more likely, or more precisely, the conditional probability of $\mathcal{E}_1$ given $\mathcal{E}_2$ is larger than or equal to the probability of $\mathcal{E}_1$:

$$P(\mathcal{E}_1 \,|\, \mathcal{E}_2) \geq P(\mathcal{E}_1). \tag{1}$$

Many interesting distributions satisfy the FKG inequality (1), among them are product measures and Gibbs measures (with some restrictions), and in particular the symmetric Bernoulli product measure from which the initial configuration of the constrained Glauber dynamics of Section 2 is chosen (corresponding to a spin system prepared at "infinite" temperature).

We will say that a spin-flip dynamics is *attractive* if, for all vertices x of $\mathbb{L}$, the rate for the spin flip $\sigma_x = -1 \to \sigma_x = +1$ is non-decreasing in the number of plus spins in σ (we will consider only symmetric dynamics, so the roles of plus and minus spins can be interchanged). Stochastic Ising models with a ferromagnetic interaction are examples of attractive dynamics. A theorem of Harris [19,20] states that attractive dynamics preserve the FKG property, i.e., if one starts with a measure P_0 that satisfies the FKG inequality and applies to the spin system an attractive dynamics, the measure P_t describing the spin system at time t still satisfies the FKG inequality. In particular, this result can be applied to the constrained Glauber dynamics of Section 2 and to the other examples in this note to deduce that the limiting (as $t \to \infty$) measure P_∞ satisfies the FKG inequality.

2 A constrained Glauber dynamics in 1D

The one-dimensional constrained Glauber dynamics studied in [10] corresponds to a ferromagnetic Ising chain where the only possible moves, happening with rate 1, are flips of plus spins surrounded by minus spins or minus spins surrounded by plus spins:

$$- + - \longrightarrow - - -, \quad + - + \longrightarrow + + +. \tag{2}$$

The blocked configurations (i.e., the absorbing states of the dynamics) are those where the unsatisfied bonds (i.e., bonds between spins of opposite sign) are isolated (see [10] for more details).

We consider the *deep-quench* situation, where the system is prepared at infinite temperature and the temperature is then decreased to zero instantaneously. This corresponds to an initial configuration chosen randomly from a symmetric Bernoulli product measure, i.e., with

$$\begin{cases} \sigma_n(0) = +1 \text{ with probability } 1/2 \\ \sigma_n(0) = -1 \text{ with probability } 1/2 \end{cases} \tag{3}$$

where $\sigma_n(t)$ is the value of the spin σ_n at time t. We call P_t the distribution of the spin configuration $\sigma(t) = \{\sigma_n(t)\}_{n\in\mathbb{Z}}$ at time t, and denote by P_∞ the limiting distribution obtained as $t \to \infty$.

2.1 Checking Edwards' hypothesis

In [10], P_∞ is compared to the uniform distribution P_{unif} on blocked configurations, corresponding to an ensemble where *all* blocked configurations have the *same* weight, using exact results on the statistics of the blocked configurations reached by the system. The comparison reveals systematic differences.

Here we confirm those results by rigorously proving that $P_\infty \neq P_{unif}$, but the main goal of this section is to introduce, via a simple specific example, a general criterion for comparing the limiting distribution P_∞ of a spin system subjected to an attractive dynamics to the uniform distribution P_{unif} on the absorbing configurations of that same dynamics. The general strategy is described in Section 3, where we also give further applications.

Consider all blocked configurations of the spin chain such that $\sigma_{\pm 2} = \sigma_{\pm 3} = +1$. It is easy to see that such blocked configurations are of only four different types:

$A \quad \sigma_{-1} = \sigma_0 = \sigma_1 = +1$
$B \quad \sigma_{-1} = +1,\ \sigma_0 = \sigma_1 = -1$
$C \quad \sigma_{-1} = \sigma_0 = -1,\ \sigma_1 = +1$
$D \quad \sigma_{-1} = \sigma_0 = \sigma_1 = -1.$

Under the uniform distribution on blocked configuration, the occurrence of each type has equal probability; therefore, conditioned on having $\sigma_{\pm 2} = \sigma_{\pm 3} = +1$,

$$\begin{aligned} &P_{unif}(\sigma_0 = +1 \,|\, \sigma_{\pm 2} = \sigma_{\pm 3} = +1) = \\ &\quad P_{unif}(A \,|\, \sigma_{\pm 2} = \sigma_{\pm 3} = +1) = 1/4. \end{aligned} \tag{4}$$

On the other hand, Harris' theorem (see Sect. 1.1) implies that P_∞ satisfies the FKG inequality, so that we have

$$P_\infty(\sigma_0 = +1 \,|\, \sigma_{\pm 2} = \sigma_{\pm 3} = +1) \geq P_\infty(\sigma_0 = +1) = 1/2, \tag{5}$$

where the equality follows from the $\pm$ symmetry of the dynamics and the initial distribution. The last two equations show that P_∞ cannot be the uniform distribution P_{unif}.

3 The general strategy

The strategy we used for the constrained Glauber dynamics of the previous section can be generalized to other attractive, symmetric Ising dynamics with locally stable configurations (later, we will also give an application where there are no locally stable configurations – see Example 4 in Sect. 3.1), with initial configuration chosen from a symmetric distribution that satisfies the FKG inequality (for instance, a product measure or a high temperature Gibbs measure). For simplicity, we restrict our attention to nearest neighbor models; in this context, by the existence of locally stable configurations we mean that there are *finite* subsets G of the lattice $\mathbb{L}$ such that, if $\sigma_x(t_0) = +1\ (-1)\ \forall x \in G$, then $\sigma_x(t) = +1\ (-1)\ \forall x \in G,\ \forall t > t_0$. When this is the case, we say that the spins in G are *stable* and we call G a *stable set*. If G is a smallest set with this property (there could be more than one, with different shapes), we call it a *minimal stable set*.

Some more notation is needed before we can proceed with the general strategy and further applications. Given a subset Λ of $\mathbb{L}$, we call *exterior boundary* $\partial_e\Lambda$ of Λ the set of vertices $x \notin \Lambda$ that are adjacent to a vertex in Λ, and *interior boundary* $\partial_i\Lambda$ of Λ the set of vertices $x \in \Lambda$ that are adjacent to a vertex not in Λ.

We are now ready to explain the general strategy; in the next section we will illustrate it with some examples. Let G_1 and G_2 be two distinct minimal stable sets both containing the origin ($0 \in G_1 \cap G_2$) and denote by $G = G_1 \cup G_2$ their union. Let L be a (finite) stable set such that $G \cap L = \emptyset$ and $\partial_e G \subset L$ (in words, G is "surrounded" by L). G_1, G_2 and L should be chosen so that $\{G \setminus G_1\} \cup L$ and $\{G \setminus G_2\} \cup L$ are stable sets. Notice that, since G_1 and G_2 are minimal stable sets, $G \setminus G_1$ and $G \setminus G_2$ are smaller than any minimal stable set and therefore are not stable sets.

Now it is easy to convince oneself that, conditioned on the spins in L all being plus, there are only four possible types of blocked configurations:

1. All the spins in G are plus.
2. All the spins in G are minus.
3. The spins in G_1 are minus and those in $G \setminus G_1$ are plus.
4. The spins in G_2 are minus and those in $G \setminus G_2$ are plus.

This implies that, conditioned on all the spins in L being plus, the uniform distribution on stable configurations assigns probability 1/4 to the event that the spin at the origin is plus (corresponding to case 1 above).

On the other hand, if we consider a symmetric, attractive dynamics with initial configuration chosen from a symmetric Bernoulli product measure, conditioned on the same event (all the spins in L being plus), the limiting distribution P_∞ must assign probability at least 1/2 to the event that the spin at the origin is plus, which shows that P_∞ cannot be the uniform distribution.

3.1 Higher dimensional examples

Here we present some examples in dimension higher than one where we can use the method described above to rule out the uniform distribution. All we have to do is choose the sets G_1, G_2 and L appropriately. We will consider zero-temperature dynamics such that a spin flips at rate 1 if it disagrees with a strict majority of its neighbors and at rate 0 otherwise. As in the example of Section 2, we will always start with a Bernoulli symmetric product measure (see (3)), corresponding to the deep-quench situation. We note that exact results are usually not available for models in dimension higher than one, which hampers the applicability of the methods used in [10].

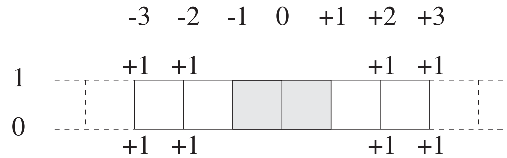

Fig. 1. The increasing event $\mathcal{E}$ used in Example 1; σ_n are the spins in the lower row $\mathbb{Z} \times \{0\}$ and σ'_n those in the upper one $\mathbb{Z} \times \{1\}$.

Example 1: *Zero-temperature dynamics on the ladder* $\mathbb{Z} \times \{0, 1\}$

The blocked configurations are such that each spin has at least two neighbors of the same sign; squares are minimal stable sets. We choose G_1 and G_2 to be the sets of vertices of the two squares containing the origin $\{0\} \times \{0\}$ (the shaded squares in Fig. 1) and L to be the set of vertices $\{\pm 2, \pm 3\} \times \{0, 1\}$.

Conditioning on the increasing event $\mathcal{E} = \{\sigma_{\pm 2} = \sigma_{\pm 3} = \sigma'_{\pm 2} = \sigma'_{\pm 3} = +1\}$ (see Fig. 1), it is easy to see that $P_{unif}(\sigma_0 = +1 \,|\, \mathcal{E}) = 1/4$, because the fact that $\sigma_0 = +1$ implies that $\sigma'_0 = \sigma_{\pm 1} = \sigma'_{\pm 1} = +1$, while there are three possible local blocked configurations with $\sigma_0 = -1$. On the other hand, $P_{unif}(\sigma_0 = +1) = 1/2$ by symmetry, so that P_{unif} does not satisfy the FKG inequality.

Example 2: *Zero-temperature dynamics on the hexagonal lattice*

The blocked configurations are again such that each spin has at least two neighbors of the same sign; hexagons are minimal stable sets. Let $\Lambda = G_1 \cup G_2 \cup L = G \cup L$ be the set of vertices of the portion of hexagonal lattice shown in Figure 2, where G_1 and G_2 are the sets of vertices of the two shaded hexagons containing the origin and $\partial_e G \subset L = \Lambda \setminus G = \partial_i \Lambda$.

Conditioning on the increasing event $\mathcal{E}$ that all the spins in $L = \partial_i\Lambda$ are +1, it is easy to see that $P_{unif}(\sigma_0 = +1 \,|\, \mathcal{E}) = 1/4$, because the fact that $\sigma_0 = +1$ implies that $\sigma_y = +1$ for all $y \in G$, while there are three possible local blocked configurations with $\sigma_0 = -1$. On the other hand, $P_{unif}(\sigma_0 = +1) = 1/2$ by symmetry, so that P_{unif} does not satisfy the FKG inequality.

Example 3: *Zero-temperature dynamics on* $\mathbb{Z}^d$

For simplicity, we consider the two-dimensional case $d = 2$, but the same reasoning works for all $d \geq 2$. In two dimensions the blocked configurations are again such that each

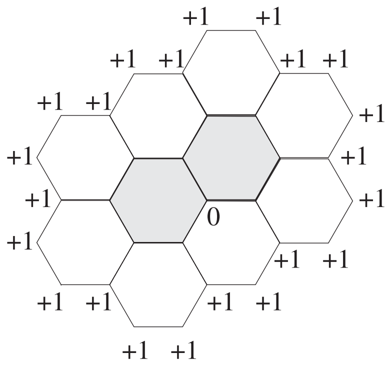

Fig. 2. The increasing event $\mathcal{E}$ used in Example 2. As indicated, the vertices of the interior boundary $\partial_i\Lambda$ of Λ all have spin +1.

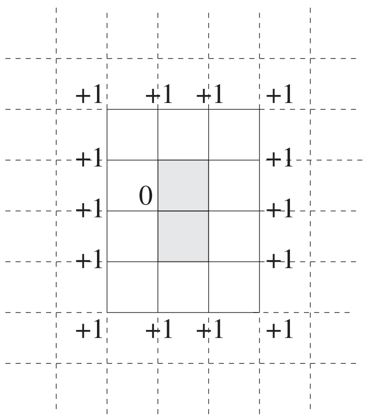

Fig. 3. The increasing event $\mathcal{E}$ used in Example 3. As indicated, the vertices of the interior boundary $\partial_i\Lambda$ of Λ all have spin +1.

spin has at least two neighbors of the same sign; squares are minimal stable sets. Let $\Lambda = G_1 \cup G_2 \cup L = G \cup L$ be the set of vertices of the portion of square lattice shown in Figure 3, where G_1 and G_2 are the sets of vertices of the two shaded squares containing the origin and $\partial_e G \subset L = \Lambda \setminus G = \partial_i\Lambda$.

Conditioning on the increasing event $\mathcal{E}$ that all the spins in $L = \partial_i\Lambda$ are +1, it is easy to see that $P_{unif}(\sigma_0 = +1 \,|\, \mathcal{E}) = 1/4$, because the fact that $\sigma_0 = +1$ implies that $\sigma_y = +1$ for all $y \in G$, while there are three possible local blocked configurations with $\sigma_0 = -1$. On the other hand, $P_{unif}(\sigma_0 = +1) = 1/2$ by symmetry, so that P_{unif} does not satisfy the FKG inequality.

Example 4: *Zero-temperature dynamics on the Cayley tree of degree* 3

This last example is interesting because, contrary to all the previous ones, there are no locally stable configurations (the only stable structures are doubly-infinite plus or minus paths). Nonetheless, the criterion described in this note can still be used.

With reference to Figure 4, conditioning on the increasing event $\mathcal{E}$ that $\sigma_{x_1} = \sigma_{x_2} = \sigma_{x_3} = +1$

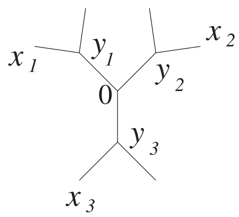

Fig. 4. A portion of the Cayley tree of degree three.

and x_1, x_2, x_3 belong to doubly-infinite +1 paths that do not contain y_1, y_2, y_3, it is easy to see that, while $P_{unif}(\sigma_0 = +1) = 1/2$ by symmetry, $P_{unif}(\sigma_0 = +1 \,|\, \mathcal{E}) = 1/5$, because if $\sigma_0 = +1$, then σ_{y_1}, σ_{y_2} and σ_{y_3} are all forced to be +1. Therefore, once again P_{unif} does not satisfy the FKG inequality.

4 Conclusions

We studied the relevance of the so called Edwards' hypothesis for certain zero-temperature spin-flip dynamics of ferromagnetic Ising models with blocked configurations. We first revisited a one-dimensional model studied in [10] and then presented new results concerning various higher-dimensional models. Although the models analyzed here are very far from the situations considered by Edwards and co-authors and for which the so called Edwards' hypothesis was proposed, they belong to a class of models where such a hypothesis could a priory hold. Recently, various authors have extensively investigated models in this class in an attempt to settle this problem (for a list of references and a nice review, see [11]). In all of the examples considered in this note, we show that the limiting distribution of the spin-flip dynamics is not the uniform distribution over blocked configurations, ruling out what in this context would appear to be a natural version of the so called Edwards' hypothesis.

Our conclusions are in line with those of several previous studies. Our approach, however, is new because it gives a criterion for testing a behavior à la Edwards which is simple and general (within the class of ferromagnetic models considered), and which allows to easily treat models in dimension higher than one. Another advantage of the method presented here is the fact that it is mathematically rigorous and thus, when applicable, it can rule out Edwards' hypothesis with absolute certainty.

The author thanks Frank den Hollander, Marco Isopi, and an anonymous referee for useful comments and acknowledges the support of a Marie Curie Intra-European Fellowship (contract MEIF-CT-2003-500740).

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